

Van der Waerden's Colouring Theorem and Weak-Strong Duality on the Lattice

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Abstract

Van der Waerden's (VDW) colouring theorem in combinatoric number theory [1] has scope for physical applications. The solution of the two colour case has enabled the construction of an explicit mapping of an infinite, one dimensional antiferromagnetic Ising system to an effective pseudo-ferromagnetic one with the coupling constants in the two cases becoming related [2]. Here the three colour problem is solved and the results are used to obtain new insights in the theory of complex lattices, particularly those relating to ternary alloys. The existence of these mappings of a multicolour lattice onto a monochromatic one with different couplings illustrates *a new form of duality*.

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One version of Van der Waerden's Colouring Theorem is [1] :

Let X be a finite subset of N^d . For each m -colouring of N^d there exists a positive integer α and a point β in N^d such that the set $\alpha X + \beta$ is monochromatic. Moreover, the number α and the coordinates of the point β are bounded by a function that depends only on X and m (and not on the particular colouring used).

Here we discuss the three colour case and show how a new type of weak-strong duality on the lattice can be realised. We first briefly review the two colour problem [2] . Next we give the solution to the three colour case and outline how this may be applicable to complex lattices and ternary alloys. We conclude by explaining the new type of weak-strong duality evident in our approach and list some more areas of application. Throughout we shall be using Ising-type nearest-neighbour interactions.

Consider the antiferromagnetic spin ordering in an infinite one dimensional lattice with lattice spacing b .

We consider N^1 and our subset X consists of any two points of the one dimensional lattice such that the first point has "up" spin (labelled by R i.e. red) and the second a "down" spin (labelled by B i.e. blue). If k denotes the point with an "up" spin then it necessarily follows that the "down" spin will occur at a spacing $k + (2m + 1)b$, $m = 0, 1, 2, \dots$. The set can be anywhere on the lattice. The coupling between any two spins is taken to be some function of their separation. To simulate the infinite lattice we use periodic boundary conditions i.e. R..B..R..B..etc. Then the VDW theorem can be quantified as

follows :

$$\begin{pmatrix} R_{\alpha k + \beta} \\ R_{\alpha[k + (2m+1)b] + \beta} \\ \alpha k + \beta \\ \alpha[k + (2m+1)b] + \beta \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{pmatrix} \begin{pmatrix} R_k \\ B_{k+(2m+1)b} \\ k \\ k + (2m+1)b \end{pmatrix} \quad (1)$$

For generality the lattice spacing in the subset X is taken to be $(2m+1)b$ ($m = 0, 1, 2, \dots$). We are considering the case where $(R..B..R..B...) \rightarrow (R..R..R..R..)$.

We have combined the colour labels (“up” and “down”) and the position labels (k and $k + (2m+1)b$) into a single column vector. The column vector on the right hand side means that there is an “up” spin (R) at position k and a “down” spin (B) at position $k + (2m+1)b$ where $(2m+1)b$ is the lattice separation. The column vector on the left hand side has the corresponding meaning. The ordering we have used in the initial configuration is (R, B). The solution to the V matrix is

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha - \beta/(2m+1)b & \beta/(2m+1)b \\ 0 & 0 & -\beta/(2m+1)b & \alpha + \beta/(2m+1)b \end{pmatrix} \quad (2)$$

Now consider the case $R \Leftrightarrow B$ in equation (1). The matrix V remains the same. This is expected as this reflects a symmetry of the system. The Van der Waerden transformation should be insensitive to what is called “up” and what is called “down”. The final configuration can also consist of all “down” spins starting with the initial ordering as (R, B). Then also the VDW matrix can be determined [2].

As the final configuration is monochromatic, it can be easily verified that for the system we are considering with periodic boundary conditions, $\alpha = 2$

and $\beta = (2m + 1)b$. In fact, it can be shown that the latter part of the theorem can also be realised viz. α and β are bounded by a function that depends only on X and m .

The energy of the initial configuration E_i may be written formally as some average of some hamiltonian $H = \sum_{ij(i \neq j)} s_i J_{ij} s_j$ with nearest neighbour interactions

$$E_i = \sum_{m, n_m \neq n} \langle m | H | n \rangle \quad (3)$$

The suggestive notation used in (3) is for convenience. To evaluate this split the chain into blocks and expand (3) in terms of these blocks. The block energies are all identical. If N be the number of blocks (3) becomes :

$$\frac{E_i}{N} = \langle u_1^T | H | u_2 \rangle = J_{21}((2m + 1)b) B_{k+(2m+1)b} R_{k+2(2m+1)b} \quad (4a)$$

where

$$|u_1 \rangle = \begin{pmatrix} R_k \\ B_{k+(2m+1)b} \\ k \\ k + (2m + 1)b \end{pmatrix}, |u_2 \rangle = \begin{pmatrix} R_{k+2(2m+1)b} \\ B_{k+3(2m+1)b} \\ k + 2(2m + 1)b \\ k + 3(2m + 1)b \end{pmatrix} \quad (4b)$$

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ J_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4c)$$

The subscripts in (4a) are only there to show that these are nearest neighbour interactions.

In a similar manner it may be shown that the final energy is

$$\frac{E_f}{N'} = \langle v_1^T | H' | v_2 \rangle$$

$$= J'_{21}(\alpha(2m+1)b)R_{\alpha[k+(2m+1)b]+\beta}R_{\alpha[k+2(2m+1)b]+\beta} \quad (5a)$$

$$|v_1 \rangle = \begin{pmatrix} R_{\alpha k+\beta} \\ R_{\alpha[k+(2m+1)b]+\beta} \\ \alpha k + \beta \\ \alpha[k+(2m+1)b] + \beta \end{pmatrix}, |v_2 \rangle = \begin{pmatrix} R_{\alpha[k+2(2m+1)b]+\beta} \\ R_{\alpha[k+3(2m+1)b]+\beta} \\ \alpha[k+2(2m+1)b] + \beta \\ \alpha[k+3(2m+1)b] + \beta \end{pmatrix} \quad (5b)$$

$$H' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ J'_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5c)$$

The antiferromagnetic chain has been mapped onto itself except that now there is an effective ferromagnetic ordering. No dynamics is involved and so the initial and final energies are equal. As usual, in the calculation of energies we take the value of the “up” spin to be +1 and that for the “down” spin to be -1. The lattice is infinite, so $N = N'$. Equating (4a) with (5a) we get :

$$J'_{21}(\alpha(2m+1)b) = -J_{21}((2m+1)b)$$

i.e.

$$J'_{21}(2(2m+1)b) = -J_{21}((2m+1)b) \quad (6)$$

Hence the crucial result of [2] was that *the antiferromagnetic system is in some sense dual to a pseudo-ferromagnetic one with a coupling opposite in sign and a function of $(\alpha)^{-1}$ times the lattice unit of the pseudo-ferromagnetic one*. The nature of this duality will become more clear in the solution to the three colour case which now follows.

Consider N^1 with the subset X consisting of any three points of the one dimensional lattice such that the first point has colour “red” (R), the second point has colour “blue” (B) and the third has colour “green” (G). If

k denotes the point with a "red" colour then it necessarily follows that the subsequent "blue" and "green" colours will occur at a spacing $k + (3m - 2)b$ and $k + 2(3m - 2)b$ respectively, $m = 1, 2, \dots$. The set can be anywhere on the lattice. The coupling between any two colours is taken to be some function of their separation. As before, we use periodic boundary conditions i.e. R..B..G..R..B..G..etc. Then the VDW theorem can be quantified as:

$$\begin{pmatrix} R_{\alpha k + \beta} \\ R_{\alpha[k + (3m - 2)b] + \beta} \\ R_{\alpha[k + 2(3m - 2)b] + \beta} \\ \alpha k + \beta \\ \alpha[k + (3m - 2)b] + \beta \\ \alpha[k + 2(3m - 2)b] + \beta \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} & V_{15} & V_{16} \\ V_{21} & V_{22} & V_{23} & V_{24} & V_{25} & V_{26} \\ V_{31} & V_{32} & V_{33} & V_{34} & V_{35} & V_{36} \\ V_{41} & V_{42} & V_{43} & V_{44} & V_{45} & V_{46} \\ V_{51} & V_{52} & V_{53} & V_{54} & V_{55} & V_{56} \\ V_{61} & V_{62} & V_{63} & V_{64} & V_{65} & V_{66} \end{pmatrix} \begin{pmatrix} R_k \\ B_{k + (3m - 2)b} \\ G_{k + 2(3m - 2)b} \\ k \\ k + (3m - 2)b \\ k + 2(3m - 2)b \end{pmatrix} \quad (7)$$

We are considering the case $(R..B..G..R..B..G..) \rightarrow (R..R..R..R..)$.

Note that for $\beta = 0$, the VDW matrix in the two colour case is an orthogonal matrix. Consistency of the formalism demands that this property should also be valid for the VDW matrix in the three colour case. Using this property, some straightforward algebra yields two solutions for the VDW matrix on right hand side of (7). These are :

$$V_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha - \beta/2(3m-2)b & 0 & \beta/2(3m-2)b \\ 0 & 0 & 0 & -\beta/2(3m-2)b & \alpha & \beta/2(3m-2)b \\ 0 & 0 & 0 & -\beta/2(3m-2)b & 0 & \alpha + \beta/2(3m-2)b \end{pmatrix} \quad (8a)$$

and

$V_2 =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (2/3)\alpha - \beta/2(3m-2)b & (2/3)\alpha & -(1/3)\alpha + \beta/2(3m-2)b \\ 0 & 0 & 0 & (2/3)\alpha - \beta/2(3m-2)b & -(1/3)\alpha & (2/3)\alpha + \beta/2(3m-2)b \\ 0 & 0 & 0 & -(1/3)\alpha - \beta/2(3m-2)b & (2/3)\alpha & (2/3)\alpha + \beta/2(3m-2)b \end{pmatrix} \quad (8b)$$

As we have periodic boundary conditions, the last lattice site at infinity is identified with the starting (first) site. So the initial energy E_i of the system can be written as :

$$\begin{aligned} E_i &= (\text{energy of primitive 3-cell}) \times N \\ &= N[R_k J_{RB}((3m-2)b) B_{k+(3m-2)b} \\ &\quad + B_{k+(3m-2)b} J_{BG}((3m-2)b) G_{k+2(3m-2)b}] \\ &\quad + G_{k+2(3m-2)b} J_{GR}((3m-2)b) R_{k+3(3m-2)b} \end{aligned}$$

i.e.

$$\begin{aligned} E_i/N &= J_{RB}((3m-2)b) R_k B_{k+(3m-2)b} \\ &\quad + J_{BG}((3m-2)b) B_{k+(3m-2)b} G_{k+2(3m-2)b} \end{aligned}$$

$$+J_{GR}((3m-2)b) \quad G_{k+2(3m-2)b} \quad R_{k+3(3m-1)b} \quad (9a)$$

Equation (9a) can be written as

$$E_i = \sum_{m, n_{m \neq n}} \langle m | H | n \rangle \quad (9b)$$

To evaluate this split the chain (as before) into blocks and expand (9b) in terms of these blocks. The block energies are all identical. If N be the number of blocks (9b) becomes :

$$\begin{aligned} E_i/N = \langle u_1^T | H | u_2 \rangle = & J_{RB}((3m-2)b) \quad R_k \quad B_{k+(3m-2)b} \\ & + J_{BG}((3m-2)b) \quad B_{k+(3m-2)b} \quad G_{k+2(3m-2)b} \\ & + J_{GR}((3m-2)b) \quad G_{k+2(3m-2)b} \quad R_{k+3(3m-2)b} \end{aligned} \quad (9c)$$

where

$$|u_1 \rangle = \begin{pmatrix} R_k \\ B_{k+(3m-2)b} \\ G_{k+2(3m-2)b} \\ k \\ k + (3m-2)b \\ k + 2(3m-2)b \end{pmatrix}, |u_2 \rangle = \begin{pmatrix} R_{k+3(3m-2)b} \\ B_{k+4(3m-2)b} \\ G_{k+5(3m-2)b} \\ k + 3(3m-2)b \\ k + 4(3m-2)b \\ k + 5(3m-2)b \end{pmatrix} \quad (10a)$$

$$H = (1/2) \begin{pmatrix} 0 & J_{BR} & J_{GR} & 0 & 0 & 0 \\ J_{RB} & 0 & J_{GB} & 0 & 0 & 0 \\ J_{RG} & J_{BG} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (10b)$$

Here we have assumed that $J_{RB} = J_{BR}$, $J_{RG} = J_{GR}$, $J_{BG} = J_{GB}$.

Similarly

$$E_f/N' = \langle v_1^T | H' | v_2 \rangle = J'_{RR}(\alpha(3m-2)b) \quad R_{\alpha(k+(3m-2)b)+\beta} \quad R_{\alpha(k+2(3m-2)b)+\beta} \quad (11a)$$

where

$$|v_1\rangle = \begin{pmatrix} R_{\alpha k + \beta} \\ R_{\alpha(k + (3m-2)b) + \beta} \\ R_{\alpha(k + 2(3m-2)b) + \beta} \\ \alpha k + \beta \\ \alpha(k + (3m-2)b) + \beta \\ \alpha(k + 2(3m-2)b) + \beta \end{pmatrix}, |v_2\rangle = \begin{pmatrix} R_{\alpha(k + 3(3m-2)b) + \beta} \\ R_{\alpha(k + 4(3m-2)b) + \beta} \\ R_{\alpha(k + 5(3m-2)b) + \beta} \\ \alpha(k + 3(3m-2)b) + \beta \\ \alpha(k + 4(3m-2)b) + \beta \\ \alpha(k + 5(3m-2)b) + \beta \end{pmatrix} \quad (11b)$$

$$H' = (1/2) \begin{pmatrix} 0 & J'_{RR} & J'_{RR} & 0 & 0 & 0 \\ J'_{RR} & 0 & J'_{RR} & 0 & 0 & 0 \\ J'_{RR} & J'_{RR} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (11c)$$

In order to calculate the energies quantitatively, let us assign the values r, c and g to the colours R, B and G respectively. These are like *weight factors* that distinguish the various colours from each other. (They are analogues of the values $+1$ and -1 in the case of "up" spin and "down" spin respectively.)

Then equating the initial and final energies and noting that for the infinite lattice $N = N'$ (as before) we have

$$\begin{aligned} r^2 J'_{RR}(\alpha(3m-2)b) &= rc J_{RB}((3m-2)b) \\ &+ cg J_{BG}((3m-2)b) \\ &+ gr J_{GR}((3m-2)b) \end{aligned} \quad (12a)$$

Two other mappings (to a monochromatic set) are also possible, *viz.*, $B..B..B..$ and $G..G..G..$. Therefore :

$$\begin{aligned} c^2 J'_{BB}(\alpha(3m-2)b) &= rc J_{RB}((3m-2)b) \\ &+ cg J_{BG}((3m-2)b) + gr J_{GR}((3m-2)b) \end{aligned} \quad (12b)$$

$$\begin{aligned}
g^2 J'_{GG}(\alpha(3m-2)b) &= rc J_{RB}((3m-2)b) \\
+cg J_{BG}((3m-2)b) &+ gr J_{GR}((3m-2)b)
\end{aligned} \tag{12c}$$

These equations imply

$$r^2/c^2 = J'_{BB}/J'_{RR}, \quad r^2/g^2 = J'_{GG}/J'_{RR}, \quad c^2/g^2 = J'_{GG}/J'_{BB} \tag{13}$$

We have thus related the couplings in the two configurations. Note that it is quite plausible that the weights r, c and g can be empirically determined. If just one of the self-couplings J'_{RR}, J'_{BB} and J'_{GG} can be estimated then the others can be determined from eq.(13). The fact that the weight factors occur quadratically in (12) and (13) allows a freedom in the sign of the couplings and so a variety of physical systems becomes amenable to treatment via our formalism. Moreover, (under certain approximations) using the two colour solution J_{RB} can be related to J'_{RR} and J'_{BB} ; J_{RG} can be related to J'_{RR} and J'_{GG} and J_{BG} can be related to J'_{BB} and J'_{GG} and so on. What we are trying to stress is that the set of equations (12) can, in principle, be solved by a judicious usage of the two-colour and three-colour solutions and some empirical or experimental inputs. Thus the entire theory of disorders can be studied in a new light. Moreover, the VDW matrices V have interesting properties which can be used to study various other aspects of complex lattices. These will be reported elsewhere.

The results of this letter bring to light a new form of duality on the lattice. A system with multiple degrees of freedom has been mapped onto a system with only a single effective degree of freedom and this is possible when the couplings are related in some particular way. Note that the lattice spacing dependence of the monocolour couplings is α times that of the multicolour couplings. Now, α is always an integer (here it works out to be 3). *Hence a*

possibility exists , depending on the exact values of the weight factors, of the monocolour coupling becoming strong (or weak) relative to some or all of the multicolour ones. This, therefore, is an example of weak-strong duality in an otherwise purely classical system.

Finally, we list some of the other possible applications of the results of this letter :

(i) In realistic situations such as in ternary alloys or in ionic (covalent) lattice in one dimension containing three ions (atoms) per unit cell the weight factors (r, c, g) may be identified with the effective charge on the ion or the number of valence electrons on the atom. J is the separation dependent bare interaction potential between near-neighbour dissimilar ions (atoms) and J' is the effective interaction potential between near-neighbour similar ions (atoms). Eqs.(12) and (13) show that a suitable combination of cations and anions can even induce an *effective attraction* between cations (anions) themselves.

(ii) Our approach based on the VDW mapping has a formal similarity to "Madelung's method" [3] of determining the cohesive energy of an ionic crystal. Besides, this approach may also be looked upon as an analogue of "dielectric function" [4] formalism at the classical level. The effective interaction calculated may be regarded as a "pseudo-potential".

(iii) The 3-colour solutions have possible applications to c -axis properties of the layered high- T_C oxide systems; 1-d or quasi-1-d organic compounds and 1-d spin glass with competing interaction and frustration.

(iv) Our approach may be generalised and slightly modified to deal with

phase separation problems.

(v) The method developed here is also useful in certain approaches to quantum gravity.

References

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